

Tree-chromatic number is not equal to path-chromatic number*

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Abstract

For a graph G and a tree-decomposition (T, \mathcal{B}) of G , the *chromatic number* of (T, \mathcal{B}) is the maximum of $\chi(G[B])$, taken over all bags $B \in \mathcal{B}$. The *tree-chromatic number* of G is the minimum chromatic number of all tree-decompositions (T, \mathcal{B}) of G . The *path-chromatic number* of G is defined analogously. In this paper, we introduce an operation that always increases the path-chromatic number of a graph. As an easy corollary of our construction, we obtain an infinite family of graphs whose path-chromatic number and tree-chromatic number are different. This settles a question of Seymour [2]. Our results also imply that the path-chromatic numbers of the Mycielski graphs are unbounded.

1 Introduction

A *tree-decomposition* of a graph G is a pair (T, \mathcal{B}) where T is a tree and $\mathcal{B} := \{B_t \mid t \in V(T)\}$ is a collection of subsets of vertices of G satisfying:

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- $V(G) = \bigcup_{t \in V(T)} B_t$,
- for each $uv \in E(G)$, there exists $t \in V(T)$ such that $u, v \in B_t$, and
- for each $v \in V(G)$, the set of all $w \in V(T)$ such that $v \in B_w$ induces a connected subtree of T .

We call each member of \mathcal{B} a *bag*. If T is a path, then we say a tree-decomposition (T, \mathcal{B}) of G is a *path-decomposition* of G . Since a path can be written as a sequence of vertices, we think of a path-decomposition of G as a sequence of sets of vertices B_1, B_2, \dots, B_s such that

- $V(G) = \bigcup_{1 \leq t \leq s} B_t$,
- for each $uv \in E(G)$, there exists $1 \leq t \leq s$ such that $u, v \in B_t$, and
- for each $v \in V(G)$, the sets B_i containing v are consecutive in the sequence.

For a tree-decomposition (T, \mathcal{B}) of G , the *chromatic number* of (T, \mathcal{B}) is $\max\{\chi(G[B_t]) \mid t \in V(T)\}$. The *tree-chromatic number* of G , denoted $\chi_T(G)$, is the minimum chromatic number taken over all tree-decompositions of G . The *path-chromatic number* of G , denoted $\chi_P(G)$, is defined analogously, where we insist that T is a path instead of an arbitrary tree. Both these parameters were introduced by Seymour [2]. Evidently, $\chi_T(G) \leq \chi_P(G) \leq \chi(G)$ for all graphs G .

The *closed neighborhood* of a set of vertices U , denoted $N[U]$, is the set of vertices with a neighbor in U , together with U itself. For every enumeration $\sigma = v_1, \dots, v_n$ of the vertices of a graph G , we denote by P_σ^G the sequence X_1, \dots, X_n of sets of vertices of G such that

$$X_\ell = N[\{v_1, \dots, v_\ell\}] \setminus \{v_1, \dots, v_{\ell-1}\}.$$

Observe that every vertex v_i of G belongs to X_i , and for $v_i v_j \in E(G)$ with $i < j$, both v_i and v_j belong to X_i . Furthermore, for $v_i \in V(G)$, if m is the first index such that $v_i \in N[\{v_m\}]$, then $v_i \in X_j$ if and only if $m \leq j \leq i$. So, P_σ^G is indeed a path decomposition of G . Let $\chi(P_\sigma^G)$ be the chromatic number of P_σ^G .

The following shows that for every graph G , there is an enumeration σ of $V(G)$ such that $\chi(P_\sigma^G) = \chi_P(G)$.

Lemma 1.1. *If G has path-chromatic number k , then there is some enumeration σ of $V(G)$ such that P_σ^G has chromatic number k .*

We prove this later in this section. Furthermore, the obvious modification of a standard dynamic programming algorithm (see Section 3 of [3]) yields a $O(n4^n)$ -time algorithm to test if G has path-chromatic number at most k .

We write $[n]$ for $\{1, 2, \dots, n\}$. For a graph G with vertex set $V(G)$, let $R_m(G)$ be the graph with vertex set $\{(i, v) \mid i \in [m], v \in V(G) \cup \{v_0\}\}$ where $v_0 \notin V(G)$, such that two distinct (i, v) and (i', v') are adjacent if and only if one of the following holds:

- $i = i'$ and exactly one of v or v' is v_0 , or
- $i \neq i'$, $v, v' \in V(G)$ and $vv' \in E(G)$.

For a subset of vertices S , we let $\langle S \rangle$ denote the subgraph induced by S (the underlying graph will always be clear). We also abbreviate $\chi(\langle S \rangle)$ by $\chi(S)$. The main theorems of this paper are the following. For an enumeration $\sigma = v_1, \dots, v_n$ of $V(G)$ with $P_\sigma^G = X_1, X_2, \dots, X_n$, we say σ is *special* if

- $\chi(P_\sigma^G) = \chi_P(G)$ and
- for every $1 \leq i \leq n$ with $\chi(X_i) = \chi_P(G)$, v_i has no neighbor in $\{v_1, \dots, v_{i-1}\}$.

Theorem 1.2. *Let n and k be positive integers, with $k \geq 2$. For every integer $m \geq n + k + 2$ and every graph G with $\chi_P(G) = k$ and $|V(G)| = n$, the path-chromatic number of $R_m(G)$ is k if there is a special enumeration of $V(G)$. Otherwise, the path-chromatic number of $R_m(G)$ is $k + 1$.*

Theorem 1.2 does not guarantee that applying R_m always increases path-chromatic number. On the other hand, our second theorem shows that applying R_m twice always increases path-chromatic number.

Theorem 1.3. *Let G be a graph with $\chi_P(G) = k$ and $|V(G)| = n$. For all integers ℓ and m such that $m \geq n + k + 2$ and $\ell \geq m(n + 1) + k + 3$, the path-chromatic number of $R_\ell(R_m(G))$ is strictly larger than k .*

Theorem 1.2 easily implies the following corollary.

Corollary 1.4. *For every positive integer k , there is an infinite family of k -connected graphs G for which $\chi_T(G) \neq \chi_P(G)$.*

These are the first known examples of graphs with differing tree-chromatic and path-chromatic numbers, which settles a question of Seymour [2]. Seymour also suspects that there is no function $f : \mathbb{N} \rightarrow \mathbb{N}$ for which $\chi_P(G) \leq f(\chi_T(G))$ for all graphs G , but unfortunately our results are not strong enough to derive this stronger conclusion.

Our results also imply that the family of Mycielski graphs have unbounded path-chromatic numbers. For $k \geq 2$, the k -Mycielski graph M_k , is the graph with $3 \cdot 2^{k-2} - 1$ vertices constructed recursively in the following way. M_2 is a single edge and M_k is obtained from M_{k-1} by adding $3 \cdot 2^{k-3}$ vertices $w, u_1, u_2, \dots, u_{3 \cdot 2^{k-3}-1}$ and adding edges wu_i for all i and $u_i v_j$ for all $i \neq j$ such that $v_i v_j \in E(M_{k-1})$ where $v_1, v_2, \dots, v_{3 \cdot 2^{k-3}-1}$ are the vertices of M_{k-1} . Here we say u_i corresponds to v_i . It is easy to show (see [1]) that for all $k \geq 2$, M_k is triangle-free and $\chi(M_k) = k$.

Corollary 1.5. *For every positive integer c , there exists a positive integer $n(c)$ such that the $n(c)$ -Mycielski graph has path-chromatic number larger than c .*

We prove Corollary 1.4 and Corollary 1.5 in Section 2. We finish this section by proving Lemma 1.1.

Proof of Lemma 1.1. For every path-decomposition $(P, \mathcal{B}) = B_1, B_2, \dots, B_s$ of G , we prove that there exists an enumeration σ of $V(G)$ such that the chromatic number of P_σ^G is at most that of (P, \mathcal{B}) . Let $\sigma = v_1, v_2, \dots, v_n$ be an enumeration of $V(G)$ such that for all $u, v \in V(G)$, if the last bag of (P, \mathcal{B}) containing u comes before the last bag of (P, \mathcal{B}) containing v then u comes before v in σ . It is easy to show that such an enumeration always exists. Let $P_\sigma^G = X_1, X_2, \dots, X_n$ and for $1 \leq i \leq n$, let $B_{\ell(i)}$ be the last bag of (P, \mathcal{B}) containing v_i . It is enough to prove that for $1 \leq i \leq n$, $B_{\ell(i)}$ contains X_i .

Suppose $v_j \in X_i \setminus B_{\ell(i)}$. Obviously $i \neq j$, and since $v_j \in X_i$, we obtain $i < j$. Let $B_{f(j)}$ be the first bag of (P, \mathcal{B}) containing v_j . Since the bags containing v_j are consecutive in (P, \mathcal{B}) , $v_j \notin B_{\ell(i)}$ and $\ell(i) \leq \ell(j)$, we obtain that $\ell(i) < f(j)$. Let v_k be a neighbour of v_j with $k \leq i$. Such a v_k exists since $v_j \in X_i$. Then, $\ell(k) \leq \ell(i)$ since $k \leq i$, so $\ell(k) < f(j)$. Therefore, there is no bag of (P, \mathcal{B}) containing both v_k and v_j because the last bag containing v_k comes before the first bag containing v_j . But this is a contradiction since $v_k v_j \in E(G)$. Thus, $X_i \subseteq B_{\ell(i)}$ as claimed, and we deduce that the chromatic number of P_σ^G is at most that of (P, \mathcal{B}) . \square

2 Deriving the Corollaries

Assuming Theorems 1.2 and 1.3, it is straightforward to derive Corollaries 1.4 and 1.5, which we do in this section. Let C_n denote the n -cycle.

Lemma 2.1. *For all odd integers $n \geq 5$ and all integers $m \geq n + 4$, the path-chromatic number of $R_m(C_n)$ is 3.*

Proof. Evidently, $\chi_P(C_n) = 2$. Hence, by Theorem 1.2, it is enough to show that every enumeration $\sigma = v_1, \dots, v_n$ of $V(C_n)$ is not special. Let $P_\sigma^G = X_1, X_2, \dots, X_n$.

Let (L, M, R) be the partition of $V(C_n)$ such that for every $v \in V(C_n)$,

- $v \in L$ if both neighbors of v come before v in σ ,
- $v \in R$ if both neighbors of v come after v in σ ,
- $v \in M$ otherwise.

Suppose M is not empty and let v_ℓ be a vertex of M . Obviously, the chromatic number of $\langle X_\ell \rangle$ is at least 2 because it contains both v_ℓ and a neighbor of v_ℓ . However, v_ℓ has a neighbor appearing before v_ℓ in σ , so σ is not special. So, we may assume M is empty. Since L and R are both stable sets, it follows that C_n is 2-colorable, a contradiction. This completes the proof. \square

On the other hand, we also have the following easy lemma.

Lemma 2.2. *For all integers $n \geq 4$ and all positive integers m , $R_m(C_n)$ has tree-chromatic number 2.*

Proof. It clearly suffices to show that $R_m(C_n)$ has tree-chromatic number at most 2. Let $V(C_n) = \{v_1, \dots, v_n\}$ with v_j adjacent to $v_{j'}$ if and only if $|j - j'| \in \{1, n - 1\}$. Let the vertex set of $R_m(C_n)$ be $\{(i, v_j) \mid i \in [m], j \in \{0\} \cup [n]\}$. We now describe a tree-decomposition (T, \mathcal{B}) of $R_m(C_n)$. Let T be a star with a center vertex c and m leaves $\ell(1), \dots, \ell(m)$. Let

- $B_c = \{(i, v_j) \mid i \in [m], j \in \{2, 3, \dots, n\}\},$
- $B_{\ell(s)} = \{(s, v_j) \mid j \in \{0, 1, 2, \dots, n\}\} \cup \{(i, v_j) \mid i \in [m], j \in \{2, n\}\}.$

We claim that (T, \mathcal{B}) is a tree-decomposition of $R_m(C_n)$ where $\mathcal{B} = \{B_t \mid t \in V(T)\}$. For $i \in [m]$ and $v_j \in V(C_n) \cup \{v_0\}$, the vertex (i, v_j) of $R_m(C_n)$ belongs to $B_{\ell(i)}$. If two distinct vertices (i, v_j) and $(i', v_{j'})$ of $R_m(C_n)$ are adjacent, then either $i = i'$ and one of v_j and $v_{j'}$ is v_0 or $i \neq i', j, j' \in [n]$ and $|j - j'| \in \{1, n - 1\}$. If the first case holds, then both vertices belong to $B_{\ell(i)}$. If the second case holds, then if either $v_j = v_1$ or $v_{j'} = v_1$ then both vertices belong to $B_{\ell(i)}$, and if neither v_j nor $v_{j'}$ is v_1 , then both belong to B_c . Lastly, for $(i, v_j) \in R_m(C_n)$, if $v_j \notin \{v_0, v_1\}$ then (i, v_j) belongs to B_c , so $\{t \mid (i, v_j) \in B_t\}$ automatically induces a subtree in T . If v_j is either v_0 or v_1 , then only $B_{\ell(i)}$ contains (i, v_j) . Hence, (T, \mathcal{B}) is a tree-decomposition, as claimed.

The set of vertices (i, v_j) of B_c with even j (or odd j) is independent. Hence, $\chi(B_c)$ is at most 2. Moreover, for $i \in [m]$, both of $\{(i, v) \mid v \in V(C_n)\}$ (note $v_0 \notin V(C_n)$) and $B_{\ell(i)} \setminus \{(i, v) \mid v \in V(C_n)\}$ are independent, so $\chi(B_{\ell(i)})$ is at most 2. We conclude that (T, \mathcal{B}) has chromatic number at most 2. This completes the proof. \square

Proof of Corollary 1.4. For every odd integer $n \geq 5$ and every integer $m \geq n + 4$, Lemma 2.1 and Lemma 2.2 show that the tree-chromatic number and path-chromatic number of $R_m(C_n)$ are different. To complete the proof, we prove that $R_m(C_n)$ is k -connected for every $n \geq k$ and $m \geq n + 4$. We prove that for every set U of vertices of $R_m(C_n)$ of size at most $k - 1$, $R_m(C_n) - U$ is connected. Again, let $V(C_n) = \{v_1, v_2, \dots, v_n\}$ with v_j adjacent to $v_{j'}$ if $|j - j'| \in \{1, n - 1\}$ and $V(R_m(C_n)) = \{(i, v_j) \mid i \in [m], j \in \{0\} \cup [n]\}$.

Since $m > |U|$, there exists $i^* \in [m]$ such that no vertex in $\{(i^*, v_j) \mid j \in \{0\} \cup [n]\}$ belongs to U . Without loss of generality, $i^* = 1$. We claim that for every vertex (i, v_j) of $R_m(C_n) - U$, there is a path from $(1, v_0)$ to (i, v_j) . We may assume that $(i, v_j) \neq (1, v_0)$. If $i = 1$, then $(1, v_0), (1, v_j)$ is a path. Hence, we may assume that $i \neq 1$. If $v_j \neq v_0$, then $(1, v_0), (1, v_{j+1}), (i, v_j)$ is a path, where $(1, v_{n+1}) = (1, v_1)$. If $v_j = v_0$, there exists $j' \in [n]$ such that $(i, v_{j'}) \notin U$ since $n > |U|$. Then $(1, v_0), (1, v_{j'+1}), (i, v_{j'}), (i, v_0)$ is a path. Therefore, $R_m(C_n) - U$ is connected. This completes the proof. \square

Recall that M_k denotes the k -Mycielski graph.

Lemma 2.3. *For all positive integers n, m and all integers $r \geq m + n$, M_r contains $R_m(M_n)$ as an induced subgraph.*

Proof. Take a sequence G_n, G_{n+1}, \dots, G_r of induced subgraphs of M_r where G_i is isomorphic to M_i and G_i is an induced subgraph of G_{i+1} for $i = n, n+1, \dots, r-1$. Let $V(G_n) = \{v_1^n, v_2^n, \dots, v_{3 \cdot 2^{n-2}-1}^n\}$, and for $s > n$, let $V(G_s) \setminus V(G_{s-1}) = \{v_0^s, v_1^s, \dots, v_{3 \cdot 2^{s-3}-1}^s\}$ where v_0^s is complete to the other vertices in this set and v_i^s corresponds to v_i^n for $1 \leq i \leq 3 \cdot 2^{n-2} - 1$. Then, the graph induced by $\{v_y^x \mid n+1 \leq x \leq m+n, 0 \leq y \leq 3 \cdot 2^{n-2} - 1\}$ is isomorphic to $R_m(M_n)$. \square

Lemma 2.3 and Theorem 1.3 together imply Corollary 1.5. Thus, it only remains to prove Theorems 1.2 and 1.3, which we do in the remaining section.

3 Proofs of Theorems 1.2 and 1.3

In this section, we prove Theorem 1.2 and Theorem 1.3. Throughout this section, G is a graph with n vertices and $R_m(G)$ has vertex set $\{(i, v) \mid i \in [m], v \in V(G) \cup \{v_0\}\}$. For $I \subseteq [m]$ and $U \subseteq V(G) \cup \{v_0\}$, we set $[I, U] = \{(i, v) \mid i \in I, v \in U\}$. We start with the following lemmas.

Lemma 3.1. *For $I \subseteq [m]$ and $U \subseteq V(G)$, suppose $|I| \geq \chi(U)$. Then there exists a map $f : U \rightarrow [I, U]$ such that*

- *for every $v \in U$, $f(v)$ belongs to $[I, \{v\}]$, and*
- *f is an isomorphism from $\langle U \rangle$ to $\langle f(U) \rangle$.*

Furthermore, for all $i^ \in [m] \setminus I$ and all $v^* \in V(G) \setminus U$, $\langle [I, U] \cup \{(i^*, v^*)\} \rangle$ contains an isomorphic copy of $\langle U \cup \{v^*\} \rangle$ as an induced subgraph.*

Proof. Let $\chi(U) = c$. Let $\mathcal{U} = (U_1, U_2, \dots, U_c)$ be a partition of U into independent sets of G . Take c distinct elements from I , say i_1, i_2, \dots, i_c , and for $v \in U$, let $f(v) = (i_s, v)$ if $v \in U_s$. We claim that f is an isomorphism from $\langle U \rangle$ to $\langle f(U) \rangle$.

Let v and v' be distinct vertices in U . If v and v' are adjacent, they are contained in distinct classes of \mathcal{U} , so $f(v)$ and $f(v')$ are adjacent by the definition of $R_m(G)$. If v and v' are non-adjacent, there are no edges between $[I, \{v\}]$ and $[I, \{v'\}]$. Hence, $f(v)$ and $f(v')$ are non-adjacent. Thus, f is an isomorphism from $\langle U \rangle$ to $\langle f(U) \rangle$.

For the last part, let $i^* \in [m] \setminus I$ and $v^* \in V(G) \setminus U$. Let f^* be the map obtained from f by adding $f^*(v^*) = (i^*, v^*)$. Since $i^* \notin I$, it easily follows that f^* is an isomorphism from $\langle U \cup \{v^*\} \rangle$ to $\langle f^*(U \cup \{v^*\}) \rangle$. This completes the proof. \square

When considering k -colorings of a graph, we always use $[k]$ for the set of colors.

Lemma 3.2. *For $I \subseteq [m]$ and $U \subseteq V(G)$, let $\chi(U) = c$. If $|I| \geq c$, the chromatic number of $\langle [I, U] \rangle$ is c . Moreover, if $|I| > c$, then for every c -coloring C of $\langle [I, U] \rangle$ and every $i \in I$, $[\{i\}, U]$ uses all c colors of C . In other words, $C([\{i\}, U]) = [c]$ for every $i \in I$.*

Proof. Let (U_1, U_2, \dots, U_c) be a partition of U into independent sets of G . Then, $([I, U_1], [I, U_2], \dots, [I, U_c])$ is a partition of $[I, U]$ and each set is independent in $\langle [I, U] \rangle$. Hence, the chromatic number of $\langle [I, U] \rangle$ is at most c . On the other hand, $\chi([I, U]) \geq c$ follows from Lemma 3.1. Thus, the chromatic number of $\langle [I, U] \rangle$ is c .

For the second part, let $C : [I, U] \rightarrow [c]$ be a c -coloring of $\langle [I, U] \rangle$. Fix $i \in I$. Since $|I \setminus i|$ is still greater than or equal to c , we can apply Lemma 3.1 to $[I \setminus i, U]$. Let f be a map from U to $[I \setminus i, U]$ as in the statement of Lemma 3.1. Let $F = f(U)$, and C_F be the restriction of C on F . As $\langle f(U) \rangle$ is not $(c-1)$ -colorable, for each color $\alpha \in [c]$, there must be a vertex $v_{\ell_\alpha} \in U$ such that $f(v_{\ell_\alpha}) \in C_F^{-1}(\alpha)$ and $f(v_{\ell_\alpha})$ has a neighbor in $C_F^{-1}(\beta)$ for every $\beta \in [c] \setminus \alpha$. Then, (i, v_{ℓ_α}) also has a neighbor in $C_F^{-1}(\beta)$ for every $\beta \in [c] \setminus \alpha$, so $C((i, v_{\ell_\alpha}))$ is α . Hence, $\{i\}, U$ sees all colors, which proves the second part. \square

Lemma 3.3. *For a graph G with path-chromatic number $k \geq 2$, let $\sigma = v_1, v_2, \dots, v_n$ be a special vertex enumeration of G . Let $P_\sigma^G = X_1, X_2, \dots, X_n$. For $j \in [n]$, if $\chi(X_j) = k$ then $\chi(X_j \setminus v_j) = k - 1$.*

Proof. It is obvious that $\chi(X_j \setminus v_j) \geq k - 1$. We may assume that $X_j \setminus v_j \neq \emptyset$. Let j' be the smallest index such that $v_{j'} \in X_j \setminus v_j$. Note that $j' > j$ and since $v_{j'}$ is contained in X_j , it has a neighbor in $\{v_1, \dots, v_j\}$. Hence, by the definition of a special vertex enumeration, $\chi(X_{j'}) \leq k - 1$. However, by the choice of j' , $X_j \setminus v_j$ is a subset of $X_{j'}$. Thus, $\chi(X_j \setminus v_j) \leq k - 1$, as required. \square

For an enumeration σ of vertices and a vertex v , let $\sigma(< v)$ denote the set of vertices which come before v in σ and $\sigma(\leq v) = \sigma(< v) \cup \{v\}$.

Lemma 3.4. *Let $m \geq n + 1$ and $\mu = (i_1, v_{j_1}), (i_2, v_{j_2}), \dots, (i_{m(n+1)}, v_{j_{m(n+1)}})$ be an enumeration of the vertices of $R_m(G)$. Let k be the chromatic number of $P_\mu^{R_m(G)}$. For each $v \in V(G)$, let $t(v)$ be the vertex in $[[m], \{v\}]$ which comes first in μ . Suppose that for all $1 \leq j < j' \leq n$, $t(v_j)$ comes before $t(v_{j'})$ in μ and let $\sigma = v_1, v_2, \dots, v_n$ be the corresponding enumeration of $V(G)$. Let $P_\sigma^G = X_1, X_2, \dots, X_n$. Then,*

- (1) *the chromatic number of P_σ^G is at most k , and*
- (2) *if $\chi(X_\ell) = k$ for some $\ell \in [n]$, then $\mu(\leq t(v_\ell))$ contains at most k vertices in $[[m], \{v_0\}]$.*

Proof. For all $v \in V(G)$, let $f(v) \in [m]$ be such that $t(v) = (f(v), v)$. Let $P_\mu^{R_m(G)} = Y_{(i_1, v_{j_1})}, Y_{(i_2, v_{j_2})}, \dots, Y_{(i_{m(n+1)}, v_{j_{m(n+1)}})}$. For the first statement, it suffices to show that for all $\ell \in [n]$, $\langle Y_{(f(\ell), v_\ell)} \rangle$ contains $\langle X_\ell \rangle$ as an induced subgraph. Let $I = [m] \setminus \{f(v_1), \dots, f(v_\ell)\}$. Then, $|I| \geq m - \ell \geq n + 1 - \ell = 1 + (n - \ell) > |X_\ell \setminus v_\ell| \geq \chi(X_\ell \setminus v_\ell)$. Moreover, $f(v_\ell) \notin I$ and $v_\ell \notin X_\ell \setminus v_\ell$. By Lemma 3.1, $\langle [I, X_\ell \setminus v_\ell] \cup \{(f(v_\ell), v_\ell)\} \rangle$ contains $\langle X_\ell \rangle$ as an induced subgraph. Since $t(v_j)$ comes before $t(v_{j'})$ in μ for all $1 \leq j < j' \leq n$, it follows that $Y_{(f(\ell), v_\ell)}$ contains $[I, X_\ell \setminus v_\ell] \cup \{(f(v_\ell), v_\ell)\}$, as required.

For the second statement, suppose $\mu(\leq t(v_\ell))$ has exactly r vertices in $[[m], \{v_0\}]$, with $r \geq k + 1$. By relabeling, we may assume that (i, v_0) is in $\mu(\leq t(v_\ell))$ for all $i \in [r]$ and that (r, v_0) appears last in μ among them. Observe that $Y_{(r, v_0)}$ contains $\{(r, v_0)\} \cup [[r], X_\ell]$. Let C be a k -coloring of $Y_{(r, v_0)}$. By Lemma 3.2, since $r > \chi(X_\ell)$, for every k -coloring of $\langle [[r], X_\ell], [\{r\}, X_\ell] \rangle$, $[\{r\}, X_\ell]$ sees all k colors. Hence $C([\{r\}, X_\ell]) = [k]$. But then there is no available color for (r, v_0) , which yields a contradiction. This completes the proof. \square

We are now ready to prove Theorem 1.2, which we restate for the reader's convenience.

Theorem 1.2. *Let n and k be positive integers, with $k \geq 2$. For every integer $m \geq n + k + 2$ and every graph G with $\chi_P(G) = k$ and $|V(G)| = n$, the path-chromatic number of $R_m(G)$ is k if there is a special enumeration of $V(G)$. Otherwise, the path-chromatic number of $R_m(G)$ is $k + 1$.*

Proof. By Lemma 3.1, $R_m(G)$ contains G as an induced subgraph, so $\chi_P(R_m(G)) \geq \chi_P(G) = k$. We break the proof up into a series of claims.

Claim 1. *If the path-chromatic number of $R_m(G)$ is k , then there exists a special enumeration of the vertices of G .*

Subproof. Let $\mu = (i_1, v_{j_1}), (i_2, v_{j_2}), \dots, (i_{m(n+1)}, v_{j_{m(n+1)}})$ be an enumeration of the vertices of $R_m(G)$ such that $P_\mu^{R_m(G)}$ has chromatic number k . Let $P_\mu^{R_m(G)} = Y_{(i_1, v_{j_1})}, Y_{(i_2, v_{j_2})}, \dots, Y_{(i_{m(n+1)}, v_{j_{m(n+1)}})}$.

For each $v \in V(G)$, let $t(v)$ be the vertex in $[[m], \{v\}]$ that appears first in μ . By renaming the vertices in G , we may assume that $t(v_j)$ comes before $t(v_{j'})$ in μ for all $1 \leq j < j' \leq n$. Let $\sigma = v_1, v_2, \dots, v_n$ be the corresponding enumeration of $V(G)$. We claim that σ is a special enumeration of $V(G)$. For each $v \in V(G)$, let $f(v) \in [m]$ be such that $t(v) = (f(v), v)$. By (1) of Lemma 3.4, P_σ^G has chromatic number at most k . Hence, $\chi(P_\sigma^G) = k$. Let $P_\sigma^G = X_1, X_2, \dots, X_n$.

Suppose σ is not special. Then, there exists $\ell \in [n]$ such that $\chi(X_\ell) = k$ and v_ℓ has a neighbor in $\{v_1, v_2, \dots, v_{\ell-1}\}$. Let $I_0 = \{i \mid (i, v_0) \in \mu(\leq t(v_\ell))\}$. By (2) of Lemma 3.4, $|I_0| \leq k$. Let $I = [m] \setminus (I_0 \cup \{f(v_1), \dots, f(v_\ell)\})$. Since $|I| \geq m - k - \ell \geq n - \ell + 2 > |X_\ell| \geq \chi(X_\ell)$, it follows that $\chi([I, X_\ell]) = k$ and for every k -coloring of $\langle [I, X_\ell], [\{i\}, X_\ell] \rangle$, $[\{i\}, X_\ell]$ sees all colors for every $i \in I$ by Lemma 3.2. Let (i, v) be the first vertex of $[I, X_\ell \cup \{v_0\}]$ that appears in μ . Either $v = v_0$ or (i, v) is adjacent to (i, v_0) . In either case, $Y_{(i, v)}$ contains $[I, X_\ell] \cup \{(i, v_0)\}$. Since $P_\mu^{R_m(G)}$ has chromatic number k , there exists a k -coloring C of $\langle Y_{(i, v)} \rangle$. Note that $C([\{i\}, X_\ell]) = [k]$. But (i, v_0) is complete to $[\{i\}, X_\ell]$, so there is no available color for (i, v_0) , a contradiction. \blacksquare

Let $\sigma = v_1, v_2, \dots, v_n$ be an enumeration of $V(G)$ with $\chi(P_\sigma^G) = k$. Let μ be the following enumeration of $V(R_m(G))$

$$(1, v_1), \dots, (m, v_1), \dots, (1, v_n), \dots, (m, v_n), (1, v_0), \dots, (m, v_0).$$

Let $P_\mu^{R_m(G)} = Y_{(1, v_1)}, Y_{(2, v_1)}, \dots, Y_{(m, v_1)}, \dots, Y_{(m, v_n)}, Y_{(1, v_0)}, \dots, Y_{(m, v_0)}$.

Claim 2. For all $i \in [m]$ and all $j \in [n]$, the chromatic number of $\langle Y_{(i,v_j)} \rangle$ is at most $\chi(X_j) + 1$.

Subproof. Suppose $\chi(X_j) = c$ and let (U_1, U_2, \dots, U_c) be a partition of X_j into independent sets of G . Observe that $Y_{(i,v_j)}$ is a subset of $[[m], X_j \cup \{v_0\}]$, and

$$[[m], X_j \cup \{v_0\}] = [[m], \{v_0\}] \cup \left(\bigcup_{p=1}^c [[m], U_p] \right).$$

Each set in the union is independent in $R_m(G)$, thus it follows that $\chi(Y_{(i,v_j)}) \leq c + 1$. \blacksquare

Claim 3. The chromatic number of $P_\mu^{R_m(G)}$ is at most $k + 1$.

Subproof. For every $i \in [m]$, $Y_{(i,v_0)}$ is a subset of $[[m], \{v_0\}]$ which is an independent set of $R_m(G)$. Hence, $\chi(Y_{(i,v_0)}) \leq 1$. By Claim 2, the chromatic number of $\langle Y_{(i,v_j)} \rangle$ is at most $k + 1$ for $i \in [m], j \in [n]$. Thus, $\chi(P_\mu^{R_m(G)}) \leq k + 1$, as required. \blacksquare

Claim 4. If σ is special, then $P_\mu^{R_m(G)}$ has chromatic number k .

Subproof. Fix $i^* \in [m]$ and $j^* \in \{0\} \cup [n]$. We will show that $\chi(Y_{(i^*,v_{j^*})}) \leq k$. If $j^* = 0$, then $\chi(Y_{(i^*,v_{j^*})}) = 1$, so may assume $j^* \neq 0$. By Claim 2, if $\chi(X_{j^*}) \leq k - 1$, then $\chi(Y_{(i^*,v_{j^*})}) \leq k$. Hence, we may assume that $\chi(X_{j^*}) = k$ and that v_{j^*} has no neighbor in $\{v_1, v_2, \dots, v_{j^*-1}\}$ by the definition of a special enumeration.

By Lemma 3.3, there is a partition $(U_1^*, U_2^*, \dots, U_{k-1}^*)$ of $X_{j^*} \setminus v_{j^*}$ into independent sets of G . For $i > i^*$, (i, v_{j^*}) has no neighbor in $\mu(< (i^*, v_{j^*}))$ since v_{j^*} has no neighbor in $\{v_1, v_2, \dots, v_{j^*-1}\}$. So, $Y_{(i^*,v_{j^*})}$ is contained in $[[m], X_{j^*} \setminus v_{j^*} \cup \{v_0\}] \cup \{(i^*, v_{j^*})\}$.

Let C be the map from $Y_{(i^*,v_{j^*})}$ to $[k]$ defined as

- for $i \neq i^*$, $C((i, v)) = s$ for all $v \in U_s^*$ and $C((i, v_0)) = k$,
- $C((i^*, v)) = k$ for all $v \in X_{j^*}$, and
- $C((i^*, v_0)) = k - 1$.

It is easy to see that C is a k -coloring of $\langle Y_{(i^*,v_{j^*})} \rangle$. Thus, $P_\mu^{R_m(G)}$ has chromatic number k , as required. \blacksquare

This last claim completes the entire proof. \square

We finish the paper by proving Theorem 1.3.

Theorem 1.3. Let G be a graph with $\chi_P(G) = k$ and $|V(G)| = n$. For all integers ℓ and m such that $m \geq n + k + 2$ and $\ell \geq m(n + 1) + k + 3$, the path-chromatic number of $R_\ell(R_m(G))$ is strictly larger than k .

Proof. Since $m \geq n + k + 2$, Theorem 1.2 shows that $R_m(G)$ has chromatic number either k or $k + 1$.

If $\chi_P(R_m(G)) = k + 1$, then since $\ell \geq m(n + 1) + k + 3 = |V(R_m(G))| + (k + 1) + 2$, the path chromatic number of $R_\ell(R_m(G))$ is either $k + 1$ or $k + 2$ which is strictly bigger than k . So, we may assume that $\chi_P(R_m(G)) = k$.

To prove that $\chi_P(R_\ell(R_m(G))) > \chi_P(R_m(G))$, it suffices to show that there is no special vertex enumeration of $R_m(G)$ by Theorem 1.2. Towards a contradiction, let $\mu = (i_1, v_{j_1}), (i_2, v_{j_2}), \dots, (i_{m(n+1)}, v_{j_{m(n+1)}})$ be a special vertex enumeration of $R_m(G)$. Let $P_\mu^{R_m(G)} = Y_{(i_1, v_{j_1})}, Y_{(i_2, v_{j_2})}, \dots, Y_{(i_{m(n+1)}, v_{j_{m(n+1)}})}$.

For each vertex v_j of G , let $t(v_j)$ be the vertex that appears first in μ among $[m], \{v_j\}$. We may assume that $t(v_j)$ comes before $t(v_{j'})$ in μ for every $1 \leq j < j' \leq n$. Let $f(v_j) \in [m]$ be such that $t(v_j) = (f(v_j), v_j)$. Let $\sigma = v_1, \dots, v_n$.

By (1) of Lemma 3.4, P_σ^G has chromatic number k . Choose $j \in [n]$ such that $\chi(X_j) = k$. We claim that $\chi(X_j \setminus v_j) = k - 1$. Let $I_0 = \{i \in [m] \mid (i, v_0) \in \mu(< t(v_j))\}$ and $I = [m] \setminus (I_0 \cup \{f(v_1), f(v_2), \dots, f(v_j)\})$. By (2) of Lemma 3.4, $|I_0| \leq k$. So, $|I| \geq m - k - j \geq (n - j + 1) + 1 > |X_j \setminus v_j| \geq \chi(X_j \setminus v_j)$. By Lemma 3.1,

$$\chi([I, X_j \setminus v_j]) \geq \chi(X_j \setminus v_j).$$

Note that $Y_{t(v_j)} \setminus t(v_j)$ contains $[I, X_j \setminus v_j]$. So, if $\chi(Y_{t(v_j)}) < k$ then $\chi([I, X_j \setminus v_j]) < k$, and if $\chi(Y_{t(v_j)}) = k$ then by Lemma 3.3, $\chi([I, X_j \setminus v_j]) < k$ as well. In either case,

$$k - 1 \geq \chi([I, X_j \setminus v_j]).$$

Combining these inequalities, we obtain $k - 1 \geq \chi(X_j \setminus v_j)$. Moreover, it is obvious that $\chi(X_j \setminus v_j) \geq k - 1$ since $\chi(X_j) = k$. Therefore, $\chi(X_j \setminus v_j) = k - 1$.

Again, as $|I| > \chi(X_j \setminus v_j)$, it follows that for every $(k - 1)$ -coloring of $\langle [I, X_j \setminus v_j] \rangle$, and every $i \in I$, $\langle \{i\}, X_j \setminus v_j \rangle$ sees all $k - 1$ colors by Lemma 3.2.

Let (i, v) be the first vertex of $[I, X_j \cup \{v_0\}]$ that appears in μ . Observe that $Y_{(i, v)}$ contains $[I, X_j \setminus v_j] \cup \{(i, v_0)\}$. For every $(k - 1)$ -coloring of $\langle [I, X_j \setminus v_j] \rangle$, $\langle \{i\}, X_j \setminus v_j \rangle$ sees all $k - 1$ colors, so $\langle [I, X_j \setminus v_j] \cup \{(i, v_0)\} \rangle$ is not $(k - 1)$ -colorable since (i, v_0) is complete to $\langle \{i\}, X_j \setminus v_j \rangle$. Thus, $\chi([I, X_j \setminus v_j] \cup \{(i, v_0)\}) = k$, and so $\chi(Y_{(i, v)}) = k$. Since μ is special, (i, v) has no neighbor in $\mu(< (i, v))$. So, v is either v_0 or v_j .

By Lemma 3.3, $\langle Y_{(i, v)} \rangle \setminus (i, v)$ is $(k - 1)$ -colorable. If $v = v_0$ then $Y_{(i, v)} \setminus (i, v)$ contains $[I, X_j]$. By Lemma 3.1, $\langle [I, X_j] \rangle$ contains $\langle X_j \rangle$ as an induced subgraph, contradicting that $\langle Y_{(i, v)} \setminus (i, v) \rangle$ is $(k - 1)$ -colorable. If $v = v_j$ then $Y_{(i, v)} \setminus (i, v)$ contains $[I, X_j \setminus v_j] \cup \{(i, v_0)\}$. Again, the chromatic number of $\langle [I, X_j \setminus v_j] \cup \{(i, v_0)\} \rangle$ is k , a contradiction.

Therefore, μ is not special. This completes the proof. \square

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